A new approach to problems of shock dynamics

Part 2. Three-dimensional problems

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This paper gives the extension of the approximate theory developed in Part 1 (Whitham 1957) to three-dimensional problems. The basic equations are derived in § 1, using the original assumption of a functional relation between the strength of the shock wave at any point and the area of the ray tube. An analogy with steady supersonic flow is found. For the diffraction of a plane shock wave by an obstacle, the equations and boundary conditions are exactly the same as those for steady supersonic potential flow past that obstacle, with a special choice of the density-speed relation. The successive positions of the shock wave are the equipotential surfaces of the supersonic flow. The 'shock-shocks' introduced in Part 1, i.e. discontinuities in the slope and Mach number of the shock wave, correspond to the steady oblique shock waves in the supersonic flow problem. They arise when Mach reflexion occurs.

In § 2 the theory is applied in detail to the diffraction of a plane shock wave by a cone. Then, in § 3, a small perturbation theory is applied to the two typical problems of (i) diffraction by a slender axi-symmetrical body of general shape, and (ii) the stability of a plane shock. Many further applications would be possible and some brief comments on these are made in § 4.

1. General theory

The theory of shock dynamics developed in Part 1 (Whitham 1957) is extended in this paper to include general three-dimensional problems. The extension is only a matter of manipulating equations; the basic assumption remains the same. We introduce the rays, which are curves orthogonal to the successive positions of the shock wave, and consider a portion of the shock wave moving along a narrow tube of neighbouring rays. Then the assumption is that the Mach number M of the shock and the area A of the ray tube are functionally related. This is suggested by the similarity of the propagation in a ray tube to the propagation of a shock wave in a tube with *solid* walls, and the function $M(A)$ is taken from the results obtained by Chisnell (1957) for that problem. The assumption and its implications have been considered in detail in Part 1. It may be added that further investigations of Chisnell's function $M(A)$ have been made since Part 1 was written; these include an alternative derivation of the formula and further discussion of its validity (see Whitham 1958).

Although the rays are not known in advance and have to be deduced as part of the solution, the relation $A = A(M)$ is sufficient to determine the motion of the shock wave without any further discussion of the dynamics of the flow. For two-dimensional problems, appropriate equations were formulated in the following way. The shock wave positions, α = constant, and the rays, β = constant, form a set of orthogonal co-ordinates (α, β) in the plane. The co-ordinate α is chosen to be $a_0 t$, where t is the time at which the shock occupies that position and a_0 is the sound speed in the undisturbed gas ahead of the shock. Then the line elements in the directions of the co-ordinate curves are $M\delta\alpha$ and $A\,\delta\beta$. For purely geometrical reasons, M and A must satisfy the relation

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{M}\frac{\partial A}{\partial \alpha}\right)+\frac{\partial}{\partial \beta}\left(\frac{1}{A}\frac{\partial M}{\partial \beta}\right)=0, \tag{1}$$

or, in more convenient form, the two relations

$$\left.\begin{aligned}\frac{\partial \theta}{\partial \beta}-\frac{1}{M}\frac{\partial A}{\partial \alpha}&=0,\\ \frac{\partial \theta}{\partial \alpha}+\frac{1}{A}\frac{\partial M}{\partial \beta}&=0,\end{aligned}\right\} \tag{2}$$

where $\theta(\alpha, \beta)$ is the angle made by the ray with a fixed direction. The dynamics assumption,

$$A = A(M), \tag{3}$$

completes a determinate set of equations for M, A, θ as functions of α and β. From the solution of these equations the motion of the shock is easily determined: for example, by integration along rays, the position of the shock at time $t = \alpha/a_0$ is given in terms of the parameter β by

$$\left.\begin{aligned}x &= x_0(\beta)+\int_0^{\alpha} M\cos\theta\,d\alpha,\\ y &= y_0(\beta)+\int_0^{\alpha} M\sin\theta\,d\alpha,\end{aligned}\right\} \tag{4}$$

where x, y are Cartesian co-ordinates, $x_0(\beta)$, $y_0(\beta)$ describe the initial shock position at $t = 0$ and θ is measured from the x-axis.

The choice of ind pendent variables based on the shock positions and the rays is particularly convenient for two-dimensional problems, and at first it was assumed that this would be true for three-dimensional problems. It seemed natural to introduce a third co-ordinate γ so that a ray would be specified by β, γ; then, relations corresponding to (1) for the line elements together with (3) would determine the motion. In fact, this procedure was rashly indicated in Part 1 as the one to be followed. It can be done, of course, but the amount of formal manipulation becomes prohibitive. The main snag is that a set of surfaces, such as the shock positions here, cannot in general be one family of a system of orthogonal co-ordinates (see, for example, Weatherburn 1931, p. 218); that is, β and γ can not be orthogonal in general. Accordingly, such a co-ordinate system becomes a positive disadvantage in three dimensions, and we revert to the Cartesian co-ordinates (x, y, z). Then the formulation becomes surprisingly simple.

The motion of the shock can be described by

$$a_0 t = \alpha(x, y, z). \tag{5}$$

and the problem now is to determine the function $\alpha(x, y, z)$. The distance δs along a ray between the shock positions at t and $t+\delta t$ is given by

$$a_0 \delta t = \delta s |\nabla \alpha|,$$

since the ray is normal to the shock. Hence, the Mach number M is given by

$$M = \frac{1}{|\nabla \alpha|}. \tag{6}$$

Next, let $\mathbf{i}(x, y, z)$ be the unit vector in the ray direction; since it is normal to the surfaces (5), it may be expressed as

$$\mathbf{i} = \frac{\nabla \alpha}{|\nabla \alpha|} = M \nabla \alpha. \tag{7}$$

Now, consider a small length of a narrow ray tube with end sections parts of surfaces $\alpha =$ constant, and let A be proportional to the cross-sectional area of the tube (measured by the area of the surface $\alpha =$ constant inside the tube at that section). Applying the divergence theorem to the vector $\mathbf{i}/A$ and the volume V inside the ray tube, we have

$$\int_V \nabla . \left(\frac{\mathbf{i}}{A}\right) d\tau = \int_S \frac{\mathbf{i} . \boldsymbol{\nu}}{A} dS, \tag{8}$$

where $\boldsymbol{\nu}$ is the outward normal to the surface S of the ray tube. On the sides of the tube, $\mathbf{i} . \boldsymbol{\nu} = 0$; on the ends, $\mathbf{i} . \boldsymbol{\nu} = \pm 1$ respectively and $\int A^{-1} dS = 1$, so that the contributions from the ends cancel. Hence the right-hand side of (8) vanishes. Therefore, since the elementary ray tube can be taken arbitrarily, the usual argument of continuity shows that

$$\nabla . \left(\frac{\mathbf{i}}{A}\right) = 0$$

everywhere. From (7), we have

$$\nabla . \left(\frac{M}{A} \nabla \alpha\right) = 0, \quad M = \frac{1}{|\nabla \alpha|}. \tag{9}$$

Since the basic assumption is that A is a known function of M, this is an equation to determine α.

A typical problem is the diffraction of an initially plane shock by a solid obstacle. If the initial Mach number of the shock is M_0 then we require

$$\alpha \sim \frac{x}{M_0} \text{ at infinity,} \tag{10}$$

and, since the shock must be normal to the surface of the obstacle, the normal derivative of α must vanish on it, i.e.

$$\frac{\partial \alpha}{\partial n} = 0 \quad \text{on the obstacle.} \tag{11}$$

For this problem the solution of (9) is required subject to the boundary conditions (10) and (11).

Analogy with steady supersonic flow

There is an immediate analogy between this problem and that of determining the velocity potential in irrotational supersonic flow past a body. For, in steady compressible flow, the continuity of mass requires

$$\nabla . (\rho \mathbf{q}) = 0, \tag{12}$$

where $\mathbf{q}$ is the velocity vector and ρ is the density. If the flow is irrotational with velocity potential ϕ then $\mathbf{q} = \nabla\phi$. Furthermore, from Bernoulli's equation, ρ is a function of the speed q; for a polytropic gas with pressure-density relation $p = \kappa\rho^\gamma$, Bernoulli's equation is

$$\frac{\gamma\kappa\rho^{\gamma-1}}{\gamma-1} + \tfrac{1}{2}q^2 = \frac{\gamma\kappa\rho_\infty^{\gamma-1}}{\gamma-1} + \tfrac{1}{2}q_\infty^2, \tag{13}$$

where ρ_∞, q_∞ refer to values in the uniform stream at infinity. Therefore, we have

$$\nabla . (\rho\nabla\phi) = 0, \tag{14}$$

where ρ is a known function of $q = |\nabla\phi|$. This corresponds exactly to (9), the precise correspondence being

$$\phi \to \alpha, \quad q \to 1/M, \quad \rho \to M/A. \tag{15}$$

The difference is that the dependence of ρ on $|\nabla\phi|$ is not the same as that of M/A on $|\nabla\alpha|$. This difference accounts for the fact that (9) is always hyperbolic, corresponding to wave motions, whereas (14) may be hyperbolic or elliptic depending upon whether the flow is supersonic or subsonic. Turning to the boundary conditions in the steady-flow problem, we require

$$\phi \sim q_\infty x \quad \text{at infinity}$$

and

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on the body.}$$

These correspond precisely to (10) and (11), so the analogy is complete.

In view of this analogy, for any problem in supersonic flow there is a corresponding one of shock diffraction which can be solved by the same method. The difference in the dependence of the functions ρ and M/A on $|\nabla\phi|$ and $|\nabla\alpha|$, respectively, will mean in general that the details of the solutions will differ. However, in the special case of a small perturbation theory, the solution can be taken over directly from the corresponding *linearized* supersonic flow with a mere change in interpretation! *Thus, any solution of a linearized supersonic flow problem solves the corresponding shock diffraction problem at the same time.* To show this we substitute

$$\alpha = \frac{1}{M_0}(x+\alpha') \tag{16}$$

in (9), and linearize the equation on the assumption that the perturbations α' are small. First,

$$M = \frac{1}{|\nabla\alpha|} = M_0 - M_0\alpha'_x + O(\alpha'^2), \tag{17}$$

and

$$\frac{M}{A} = \left[\frac{M}{A}\right]_{M=M_0} + \left[\frac{d}{dM}\left(\frac{M}{A}\right)\right]_{M=M_0}(-M_0\alpha'_x) + O(\alpha'^2)$$

$$= \left[\frac{M}{A}\right]_{M=M_0} - \left[\frac{M}{A} - \frac{M^2}{A^2}\frac{dA}{dM}\right]_{M=M_0}\alpha'_x + O(\alpha'^2). \tag{18}$$

Then, substituting (16) and (18) in (9), we have

$$\alpha'_{yy} + \alpha'_{zz} - B^2 \alpha'_{xx} = 0, \quad B^2 = \left[-\frac{M}{A}\frac{dA}{dM}\right]_{M=M_0}. \tag{19}$$

In supersonic flow, the velocity potential is taken in the form

$$\phi = q_\infty(x + \phi'), \tag{20}$$

and the linearized equation satisfied by ϕ' is found to be

$$\phi'_{yy} + \phi'_{zz} - B^2 \phi'_{xx} = 0, \quad B^2 = \mathscr{M}_\infty^2 - 1, \tag{21}$$

where $\mathscr{M}_\infty$ is the Mach number of the stream at infinity. At infinity, α' and ϕ' tend to zero, and on the obstacle the normal derivatives of $x + \alpha'$ and $x + \phi'$ must both vanish. Therefore, the problems to be solved for α' and ϕ' are exactly the same. The interpretations of the solution are different, and the coefficients B have different definitions in terms of the given parameters. *The equipotentials of the supersonic flow problem are the shock positions of the diffraction problem; the streamlines are the rays.*

Solutions obtained by use of this analogy will be presented in § 3.

Two-dimensional problems

The analogy with supersonic flow also provides a useful way of comparing the formulation (9) with (2) for two-dimensional problems. As noted above, (9) corresponds to the equation for ϕ in Cartesian co-ordinates (x, y). Now, (2) corresponds to the equations for speed q and flow direction θ as functions of the velocity potential ϕ and stream function ψ. The latter equations are

$$\left.\begin{aligned} \frac{\partial \theta}{\partial \phi} &= \frac{\rho}{q}\frac{\partial q}{\partial \psi}, \\ \frac{\partial \theta}{\partial \psi} &= q\frac{\partial}{\partial \phi}\left(\frac{1}{\rho q}\right); \end{aligned}\right\} \tag{22}$$

the correspondence with (2) is

$$\phi \to \alpha, \quad \psi \to \beta, \quad \theta \to \theta, \quad q \to 1/M, \quad \rho \to M/A, \tag{23}$$

in agreement with (15). (Equations (22) can be derived from the geometry of equipotentials and streamlines in exactly the same way that equations (2) were obtained in Part 1 from the geometry of shocks and rays.)

Equations (2) are more convenient than (9) for the problems of Part 1, just as, for many purposes, equations (22) are more convenient than (12) for two-dimensional flow. This is particularly true in finding the characteristic form of the equations. It was found in Part 1, that the characteristic form of equations (2) is

$$\theta \pm \int \frac{dM}{Ac} = \text{constant} \tag{24}$$

on curves

$$\frac{d\beta}{d\alpha} = \pm c, \tag{25}$$

where $c(M)$, the propagation speed in the (α, β) co-ordinates, is given by

$$c(M) = \sqrt{\left(-\frac{M}{A}\frac{dM}{dA}\right)}. \tag{26}$$

(See equations (10), (11) and (12) of Part 1.) The variables M and θ are given in terms of the derivatives of α by

$$\alpha_x^2 + \alpha_y^2 = 1/M^2, \quad \alpha_y/\alpha_x = \tan\theta, \tag{27}$$

which allows the Riemann invariants $\theta \pm \int dM/Ac$ to be written in terms of α_x, α_y. To write the slopes of the characteristics in terms of x and y, we note that

$$\frac{dy}{dx} = \frac{y_\alpha + y_\beta(d\beta/d\alpha)}{x_\alpha + x_\beta(d\beta/d\alpha)},$$

and $$x_\alpha = M\cos\theta, \quad y_\alpha = M\sin\theta, \quad x_\beta = -A\sin\theta, \quad y_\beta = A\cos\theta. \tag{28}$$

Therefore, the characteristics $d\beta/d\alpha = \pm c$ become

$$\frac{dy}{dx} = \tan(\theta \pm m) \quad \text{where} \quad \tan m = \frac{Ac}{M} = \sqrt{\left(-\frac{A}{M}\frac{dM}{dA}\right)}. \tag{29}$$

The angle m corresponds to the Mach angle in supersonic flow. It is observed that the coefficient B which appears in the linearized equation (19) is $\cot m_0$, where m_0 is the angle of the characteristic in the undisturbed region ahead of the obstacle, where $\theta = 0$, $M = M_0$. Equations (24), (27) and (29) provide the characteristic form of (9) in two-dimensional problems. Clearly, even when using the (x, y) plane, it is often convenient to work with M and θ as dependent variables.

The simplest problem treated in Part 1 was the motion of an initially plane shock along a curved wall. It will be useful, perhaps, to indicate how the solution is obtained directly in the (x, y) plane. By the usual argument, the solution must be a simple wave with

$$\theta = \int_{M_0}^{M} \frac{dM}{Ac} \tag{30}$$

holding throughout the flow (since all the negative characteristics start in the undisturbed region where $M = M_0$, $\theta = 0$). This gives the Mach number M_w at the wall in terms of the angle θ_w of the wall without further calculation. In the complete solution, θ and M remain constant on each positive characteristic; as a consequence these characteristics are straight lines. The solution may be written in terms of a characteristic parameter ξ as

$$\theta = \theta_w(\xi), \quad M = M_w(\xi), \tag{31}$$

$$y = y_w(\xi) + (x - \xi)\tan\{\theta_w(\xi) + m_w(\xi)\}, \tag{32}$$

where $y = y_w(x)$ describes the shape of the wall, $\theta_w(x) = \tan^{-1} y_w'(x)$, $M_w(x)$ is found from (30) and $m_w(x)$ is the corresponding value of m. If the wall turns away from the flow region, the solution is an 'expansion wave' with diverging characteristics as shown in figure 1. If the wall turns into the flow region a 'compression wave' results and the characteristics converge as shown in figure 2. The shock positions are lines of constant α. Since $\alpha_x = \cos\theta/M$, α is given by

$$\alpha = \int_{x_0}^{x} \frac{\cos\theta}{M}\,dx, \tag{33}$$

where $x = x_0$ is the initial position of the shock at $t = 0$; the curve $\alpha = a_0 t$ gives the position at time t. Shock positions are indicated by the broken lines in figures 1 and 2; it should be remembered that they correspond to the equipotentials of supersonic flow. The characteristics in the (x, y) plane give the locus of the waves as they move along the shock.

In the case of a compression wave, the characteristics form an envelope and the continuous solution given in (31) and (32) breaks down. This corresponds precisely to the 'breaking' of the waves propagating along the shock and the formation of a *shock-shock* as described in Part 1. The formation of a shock-shock indicates Mach reflexion as explained in Part 1. In the (x, y)-plane the shock-shock will be represented as a curve across which M and θ are discontinuous. This curve will be the locus of the shock-shock as it moves along the shock wave;

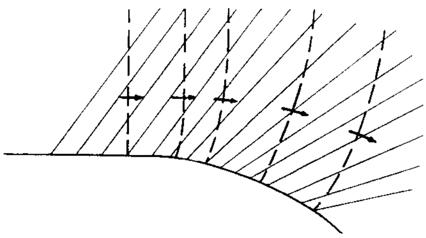

FIGURE 1

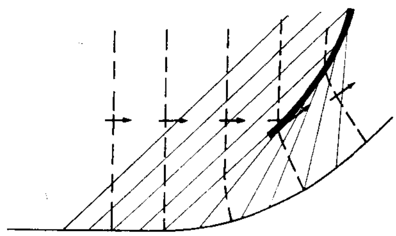

FIGURE 2

FIGURE 1. Motion of a shock wave along a convex wall; the full lines are characteristics and the broken lines are successive shock-wave positions.

FIGURE 2. Motion of a shock wave along a concave wall.

although there is a slight difference in sense, this locus will simply be called a shock-shock. It corresponds to the steady oblique shock which appears in similar circumstances in steady supersonic flow. It is shown as a thicker line in figure 2. The relations connecting the values of M and θ on the two sides of a shock-shock with its velocity have been established in Part 1. Here, they are required in the form of relations to be satisfied across the shock-shock curve in the (x, y) plane. In addition, the general form of the relations for three-dimensional problems must be established.

Shock-shock relations

First of all, since the portions of the shock wave must be connected, α must be continuous across the shock-shock. Hence, it follows that the tangential derivatives of α on the two sides of the shock-shock must be equal. If $\mathbf{n}$ is the unit vector normal to the surface of the shock-shock, this condition may be written

$$\mathbf{n} \times (\nabla\alpha)_0 = \mathbf{n} \times (\nabla\alpha)_1, \tag{34}$$

where subscripts 0 and 1 denote values on the two sides of the shock-shock. The other condition concerns the jump in the normal derivative of α, and it may be found as follows. Consider a narrow ray tube which intersects a small portion of the shock-shock surface. If $\mathbf{i}_0$, $\mathbf{i}_1$ are the directions of the ray on the two sides and A_0, A_1 are corresponding cross-sectional areas of the ray tube, it is clear that the area intercepted on the shock-shock has a projection A_0 in the $\mathbf{i}_0$ direction and a projection A_1 in the $\mathbf{i}_1$ direction. Therefore,

$$\frac{\mathbf{n}.\mathbf{i}_0}{A_0} = \frac{\mathbf{n}.\mathbf{i}_1}{A_1}.$$

In terms of α, this condition may be written

$$\frac{M_0}{A_0}\mathbf{n}.(\nabla\alpha)_0 = \frac{M_1}{A_1}\mathbf{n}.(\nabla\alpha)_1. \tag{35}$$

It may be remarked that (9) shows that the flux of $M\nabla\alpha/A$ through a closed surface is zero in regions where $M\nabla\alpha/A$ is continuous; equation (35) shows that the flux of $M\nabla\alpha/A$ is conserved even across discontinuities. This gives a quick

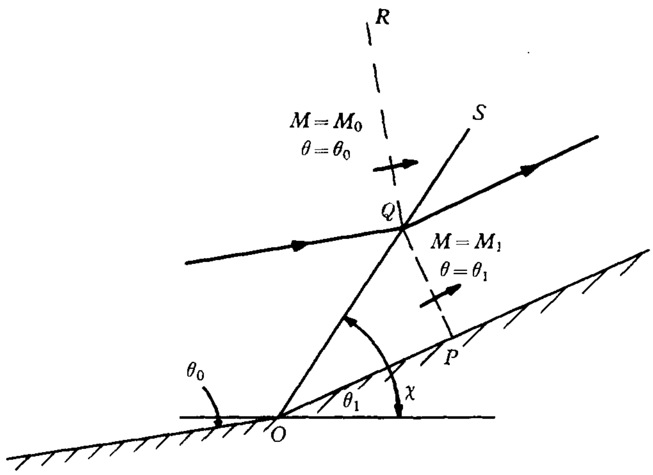

FIGURE 3. Shock-shock at a concave corner; OS is the shock-shock and PQR represents a typical shock-wave position.

way of guessing the appropriate shock condition from an equation of motion in divergence form, but it must always be verified independently that the quantity is indeed conserved across a discontinuity. For example, in continuous regions of a supersonic flow the entropy S is conserved, i.e.

$$\nabla.(\rho\mathbf{q}S) = 0,$$

but it is *not* true that the normal flux of entropy on the two sides of a shock are the same.

Equations (34) and (35) constitute the shock-shock relations for the three-dimensional problem. Remembering that $\nabla\alpha$ corresponds to $\mathbf{q}$ and M/A corresponds to ρ in steady supersonic flow, we see that (34) corresponds to the continuity of tangential velocity across an oblique shock wave and (35) corresponds to the conservation of mass flux. In supersonic flow a further relation is added (involving the normal component of momentum) since the entropy changes across the shock wave and so the function $\rho(q)$ is modified. The extra relation is required to determine this modification. In the shock dynamics, this would correspond to a change in the functional dependence of A on M and it will be ignored; but it is an important point and is discussed in detail in Part 1, pp. 155–6. It corresponds to ignoring the entropy changes in supersonic flow and continuing to assume potential flow behind a shock wave.

It may be verified that (34) and (35) reduce to the relations found in Part 1 for the two-dimensional case. In that case, we may consider the oblique shock-shock separating the two uniform regions as shown in figure 3. The shock-shock is the line OS and a typical position of the shock wave is PQR. (This shows the Mach reflexion of the shock wave when it is diffracted by a wedge.) If the angle of the shock-shock to the x-axis is χ, then remembering that $\nabla\alpha = \mathbf{i}/M$, (34) and (35) give

$$\frac{\cos(\chi-\theta_0)}{M_0} = \frac{\cos(\chi-\theta_1)}{M_1}, \tag{36}$$

$$\frac{\sin(\chi-\theta_0)}{A_0} = \frac{\sin(\chi-\theta_1)}{A_1}. \tag{37}$$

These relations can be solved to give θ_1 and χ in terms of M_0, θ_0, M_1 as follows:

$$\tan(\theta_1-\theta_0) = \frac{(M_1^2-M_0^2)^{\frac{1}{2}}(A_0^2-A_1^2)^{\frac{1}{2}}}{A_1M_1+A_0M_0}, \tag{38}$$

$$\tan(\chi-\theta_0) = \frac{A_0}{M_0}\left(\frac{M_1^2-M_0^2}{A_0^2-A_1^2}\right)^{\frac{1}{2}}. \tag{39}$$

These agree with the form found in Part 1; (39) corresponds to the relation

$$C = \left(\frac{M_1^2-M_0^2}{A_0^2-A_1^2}\right)^{\frac{1}{2}}$$

for the shock-shock velocity C in the (α, β) co-ordinates.

The A-M relation

Before proceeding to applications of this basic theory to specific problems, it is convenient to repeat the main properties of the function $A(M)$ for purposes of reference. All the information has been given in Part 1. The function $A(M)$ may be written

$$A = A_0\exp\left\{-\int_{M_0}^{M}\frac{2M}{(M^2-1)K(M)}\,dM\right\}, \tag{40}$$

where

$$K(M) = 2\left[\left(1+\frac{2}{\gamma+1}\frac{1-\mu^2}{\mu}\right)(2\mu+1+M^{-2})\right]^{-1}, \quad \mu^2 = \frac{(\gamma-1)M^2+2}{2\gamma M^2-(\gamma-1)}. \tag{41}$$

The function $K(M)$ decreases slowly from 0·5 at $M = 1$ to 0·3941 (for $\gamma = 1{\cdot}4$) as $M \to \infty$. Thus, for weak shocks,

$$\frac{A}{A_0} \sim \left(\frac{M_0-1}{M-1}\right)^2, \quad M-1 \ll 1, \tag{42}$$

and for strong shocks

$$\frac{A}{A_0} \sim \left(\frac{M_0}{M}\right)^n, \quad n = \frac{2}{K(\infty)} = 5{\cdot}0743 \quad (M \gg 1). \tag{43}$$

The strong shock formula is particularly useful because it simplifies all the expressions, and also it covers the most important range of shock strengths. It becomes a good approximation for $M > 3$, say. It should also be noted that when this approximation applies, the expressions for α/M_0, M/M_0, θ in any problem will be independent of M_0. Thus, a single calculation for a given geometry is sufficient for all M_0. *The shock wave will go through the same sequence of positions for all M_0; only the time scale will be modified.*

A graph of the function $\int_1^M dM/Ac$, which appears in the Riemann invariants (24), has been given in Part 1. We note the limiting values

$$\int_{M_0}^{M} \frac{dM}{Ac} \sim 2^{\frac{3}{2}}\{(M-1)^{\frac{1}{2}}-(M_0-1)^{\frac{1}{2}}\} \quad (M-1 \ll 1), \tag{44}$$

$$\int_{M_0}^{M} \frac{dM}{Ac} \sim n^{\frac{1}{2}} \log \frac{M}{M_0} \quad (M \gg 1). \tag{45}$$

The angle m between the characteristic direction and the ray direction can be determined from (29) and (40). It is found to be given by

$$\tan m = \left\{\frac{(M^2-1)\,K(M)}{2M^2}\right\}^{\frac{1}{2}}. \tag{46}$$

For weak shocks,

$$\tan m \sim \left(\frac{M-1}{2}\right)^{\frac{1}{2}} \quad (M-1 \ll 1); \tag{47}$$

for strong shocks,

$$\tan m \to n^{-\frac{1}{2}} = 0{\cdot}4439, \quad m \to 23{\cdot}9^\circ, \quad M \to \infty. \tag{48}$$

The coefficient B that appears in the linearized equation (19) is just $\cot m_0$; hence, $B \to n^{\frac{1}{2}} = 2{\cdot}253$ as $M \to \infty$.

2. Diffraction by a cone

The simplest three-dimensional problem is the diffraction of a plane shock wave by a cone; this corresponds to the well-known Taylor-Maccoll problem of flow past a cone in supersonic flow. For axi-symmetrical problems such as this, it is still possible to use independent variables based on the shock positions and rays. The only modification of (2) would be the replacement of A by A/r, where x is the distance from the axis of symmetry, and the addition of the equation

$$\frac{\partial r}{\partial \alpha} = M \cos\theta$$

to determine r. But, it will be more convenient to work directly from (9) in the (x, r) plane.

In terms of M and θ,

$$\alpha_x = \frac{\cos\theta}{M}, \quad \alpha_r = \frac{\sin\theta}{M}, \tag{49}$$

so that

$$\frac{\partial}{\partial x}\left(\frac{\sin\theta}{M}\right) - \frac{\partial}{\partial r}\left(\frac{\cos\theta}{M}\right) = 0, \tag{50}$$

and, from (9),

$$\frac{\partial}{\partial x}\left(\frac{r\cos\theta}{A}\right)+\frac{\partial}{\partial r}\left(\frac{r\sin\theta}{A}\right)=0. \tag{51}$$

In the problem of diffraction by a cone, the only parameters prescribed in the problem are the initial Mach number M_0 and the cone angle θ_w; there is no length. Hence in the solution, M and θ must be functions of the single variable $\eta = \tan^{-1} r/x$. Then, (50) and (51) become

$$\frac{1}{M}\frac{dM}{d\eta} = \tan(\eta-\theta)\frac{d\theta}{d\eta}, \tag{52}$$

$$-\frac{1}{A}\frac{dA}{d\eta} = \left(-\frac{M}{A}\frac{dA}{dM}\right)\frac{1}{M}\frac{dM}{d\eta} = \left\{\frac{d\theta}{d\eta}+\frac{\tan\theta}{\sin\eta\,\cos\eta\,(1+\tan\eta\,\tan\theta)}\right\}\cot(\eta-\theta). \tag{53}$$

The disturbed region is separated from the uniform region in which $M = M_0$, $\theta = 0$ by a shock-shock at which η is equal to the shock-shock angle χ and the relations (38), (39) must be satisfied. So the solution of (52) and (53) is required such that

$$\left.\begin{aligned} \theta &= \theta_1, \quad M = M_1 \\ \tan\theta_1 &= \frac{(M_1^2-M_0^2)^{\frac{1}{2}}(A_0^2-A_1^2)^{\frac{1}{2}}}{A_1M_1+A_0M_0} \\ \tan\chi &= \frac{A_0}{M_0}\left(\frac{M_1^2-M_0^2}{A_0^2-A_1^2}\right)^{\frac{1}{2}} \end{aligned}\right\} \quad \text{at } \eta = \chi, \tag{54}$$

and

$$\theta = \theta_w \text{ at the cone } \eta = \theta_w. \tag{55}$$

These are essentially two boundary conditions since χ is not known in advance. The procedure for finding solutions for given M_0 is to choose a value for M_1, then χ and θ_1 can be found from (54) and the pair of equations (52) and (53) integrated from $\eta = \chi$ down to the point where it is found that $\theta = \eta$. This point must be the surface of the cone and it gives the value of the cone angle θ_w which corresponds to the chosen value of M_1.

If the strength of the shock wave is fairly large ($M_0 > 3$, say) the asymptotic form (43) may be used for the A-M relation and there is considerable simplification in the whole calculation. Most important of all, only the Mach number ratio M/M_0 is significant, and one calculation gives the solution for a given cone *for all* M_0. We set $M/M_0 = R$, then $A/A_0 = R^{-n}$ and

$$-\frac{M}{A}\frac{dA}{dM} = n.$$

Hence, a single equation for $\theta(\eta)$ can be obtained from (52) and (53). We have

$$\frac{d\theta}{d\eta} = \frac{\tan\theta}{\sin\eta\,\cos\eta\,(1+\tan\eta\,\tan\theta)\{n\tan^2(\eta-\theta)-1\}} \tag{56}$$

and

$$\frac{1}{R}\frac{dR}{d\eta} = \tan(\eta-\theta)\frac{d\theta}{d\eta}. \tag{57}$$

The shock-shock relations become

$$\left.\begin{aligned} \tan\theta_1 &= \frac{(R_1^2-1)^{\frac{1}{2}}(1-R_1^{-2n})^{\frac{1}{2}}}{1+R_1^{1-n}} \\ \tan\chi &= \left(\frac{R_1^2-1}{1-R_1^{-2n}}\right)^{\frac{1}{2}} \end{aligned}\right\} \quad \text{at } \eta = \chi. \tag{58}$$

To solve the problem a value of R_1 is chosen, the starting value of θ at $\eta = \chi$ is found from (58), and (56) is integrated to the point where $\theta = \eta$. This common value is the cone angle θ_w. The Mach number ratio R can then be found from an integration of (57).

The position of the shock wave at any time can be found from the solution for $\theta(\eta)$, $R(\eta)$ without any further integration. For α takes the form $xf(\eta)/M_0$, so

$$\alpha_x = \frac{1}{M_0}\{f(\eta)-\sin^2\eta f'(\eta)\} = \frac{\cos\theta}{R},$$

$$\alpha_r = \frac{1}{M_0}\sin\eta\cos\eta f'(\eta) = \frac{\sin\theta}{R}.$$

Therefore,
$$f(\eta) = \frac{\cos\theta+\sin\theta\tan\eta}{R}. \tag{59}$$

Hence, at time t after the shock strikes the vertex of the cone, $\alpha = a_0 t$ and

$$\left.\begin{aligned} \frac{x}{U_0 t} &= \frac{1}{f(\eta)} = \frac{R}{\cos\theta+\sin\theta\tan\eta}, \\ \frac{r}{U_0 t} &= \frac{x}{U_0 t}\tan\eta, \end{aligned}\right\} \tag{60}$$

gives the position of the shock wave in terms of the parameter η. It should be especially noted that x/U_0t, r/U_0t, the shock-shock angle χ and the distributions of θ and M/M_0 with η are all independent of M_0. *Thus, for a given cone, all shock waves go through exactly the same sequence of positions*; the differences in U_0 affect only the time scale.

The calculation has been carried out* for $R_1 = M_1/M_0 = 1{\cdot}2$, corresponding to $\theta_1 = 22{\cdot}4^\circ$ and a shock-shock angle $\chi = 35{\cdot}8^\circ$. The corresponding cone is found to be $\theta_w = 28{\cdot}8^\circ$. The change in R is very small; it rises only to $1{\cdot}216$ at the cone. The position of the shock is shown in figure 4, and the distributions of θ and M/M_0 with η are given in figures 5 and 6. It should be remarked that since θ and R do not change very much, this solution is quite close to the corresponding two-dimensional one of diffraction by a wedge in which θ and M/M_0 remain constant. As $R_1 = M_1/M_0$ increases further, θ_w increases and the changes in θ and M/M_0 between the shock-shock and the cone get smaller. At the same time, the angle between the shock-shock and the cone decreases (from (58), $\chi \to \theta_1$ as $R_1 \to \infty$). As explained in Part 1 for the wedge problem, there is no critical angle at which Mach reflexion goes over into regular reflexion when the simple dependence of A on M is used across shock-shocks.

* The author is indebted to Mr J. Engelhardt for this calculation.

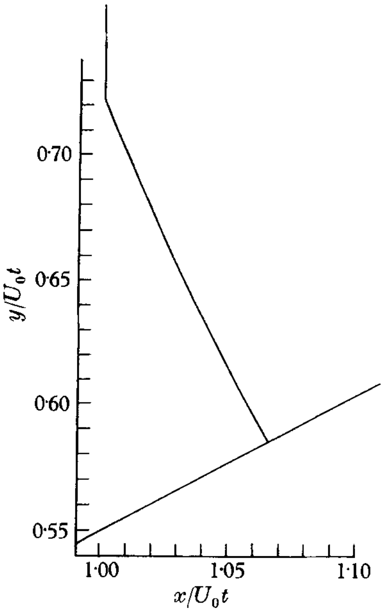

FIGURE 4. Calculated position of shock wave in diffraction by a cone.

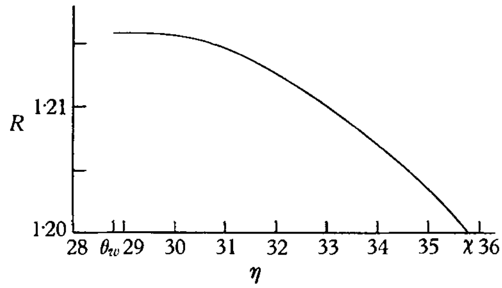

FIGURE 5. Distribution of $R = M/M_0$ with $\eta = \tan^{-1} r/x$ for shock wave in figure 4.

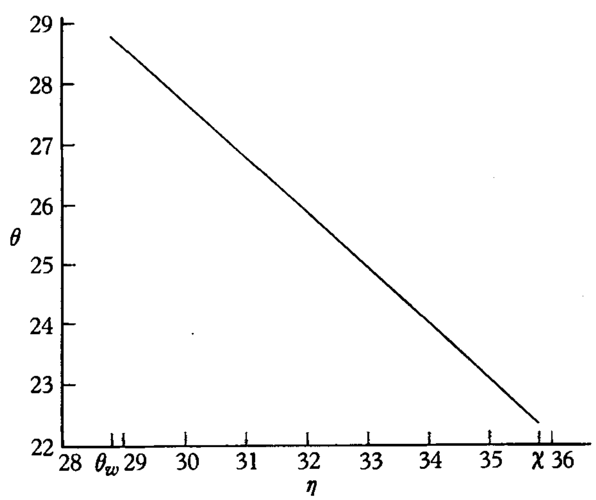

FIGURE 6. Distribution of θ with $\eta = \tan^{-1} r/x$ for shock wave in figure 4.

As R_1 decreases, the solution becomes less like the two-dimensional. When $R_1 - 1$ is small, corresponding to a slender cone with small angle θ_w, the changes in M/M_0 and θ across the shock-shock are of much smaller order in θ_w than the changes between the shock-shock and the cone surface. An analytic expression for the solution can be found in this case; it is included in the next section where the general solution is found for a slender axi-symmetrical obstacle of arbitrary shape. The jumps in M/M_0 and θ across the shock-shock are found to be

$$\left.\begin{aligned} \frac{M_1}{M_0} - 1 &= 18{\cdot}2\theta_w^4, \\ \theta_1 &= 41{\cdot}0\theta_w^4, \end{aligned}\right\} \tag{61}$$

whereas at the cone $\theta = \theta_w$ and

$$\frac{M_w}{M_0} - 1 = \theta_w^2\left\{\log\frac{1}{\theta_w} - 0{\cdot}619\right\}. \tag{62}$$

Finally, it should be remarked that the shock-shock represents Mach reflexion, but the reflected shock wave is suppressed in this work. However, the strength and angle of the reflected shock wave at the triple point can be calculated from the values of θ_1 and M_1 using the shock-wave relations.

3. Small perturbation problems

For perturbations to a plane shock wave moving in the x-direction with Mach number M_0,

$$\alpha = \frac{1}{M_0}(x + \alpha'),$$

where α'/x is small. The linearized equation to be satisfied by α' is the wave equation

$$\alpha'_{yy} + \alpha'_{zz} - B^2\alpha'_{xx} = 0$$

as found in (19). There are two main types of perturbation problem: (i) diffraction by thin or slender obstacles, and (ii) stability problems in which the initial shape and strength of the shock wave are given. An example of each is now considered.

(i) *Diffraction by a slender body*

If the perturbations are produced by a thin or slender obstacle, the boundary conditions are $\partial(x + \alpha')/\partial n = 0$ on the obstacle, and $\alpha' = 0$ at infinity. As explained already the solutions can be taken over from the theory of linearized supersonic flow. There is no point in copying down a long list of these. One example will be discussed here and the full range of possibilities can be seen by looking through a standard book on supersonic flow (for example, Ward (1955) or Sears (1955)).

It is worth noting in some detail the important special case of diffraction by an axi-symmetrical slender body. If the cross-sectional area of the body at a distance x from the nose is $S(x)$, the solution of (19), satisfying the boundary conditions, is

$$\alpha' = -\frac{1}{2\pi}\int_0^{x-Br}\frac{S'(\xi)}{\sqrt{[(x-\xi)^2 - B^2r^2]}}\,d\xi. \tag{63}$$

(This assumes that $S'(x)$ is continuous.) In deriving M and θ from this solution it is important to notice that α'_x and α'_r are not of the same order near the body. If the maximum slope δ of the surface of the body is taken as a measure of the slenderness, $\alpha'_r \propto \delta$, while $\alpha'_x \propto \delta^2 \log 1/\delta$ near the body. Therefore, in calculating M, the approximate expression

$$\frac{M-M_0}{M_0} = -\alpha'_x - \tfrac{1}{2}\alpha'^2_r + O\left(\delta^4 \log^2 \frac{1}{\delta}\right) \tag{64}$$

should be used, instead of the simple linearized expression (17). This appears to be inconsistent since (17) was used in deriving (19), but the corresponding procedure has been fully investigated in supersonic flow and shown to give valid approximations. There is no trouble with θ; it is simply given by α'_r. Away from the body, both α'_x and α'_r are $O(\delta^2)$ and the α'_r term in (64) can be dropped.

The linearized solution does not include an adequate description of the shock-shock. Due to the linearization, the shock-shock degenerates into the characteristic $x - Br = 0$, but an inspection of (63) shows that α', α'_x, α'_r are continuous there. When $(x-Br)/r$ is small, the expressions for α'_x and α'_r may be approximated as

$$\left.\begin{aligned} \alpha'_x &= -\frac{M-M_0}{M_0} = -\frac{F(x-Br)}{\sqrt{(2Br)}}, \\ \alpha'_r &= \theta = \frac{BF(x-Br)}{\sqrt{(2Br)}}, \end{aligned}\right\} \tag{65}$$

where
$$F(x) = \frac{1}{2\pi}\int_0^x \frac{S''(\xi)\,d\xi}{\sqrt{(x-\xi)}}. \tag{66}$$

Since $F(x) \to 0$ as $x \to 0$, $M - M_0$ and θ vanish at $x - Br = 0$ and so there is no shock-shock. This result arises because the true values of $M - M_0$ and θ at the shock-shock are of smaller order in δ than the typical ones and they get neglected in the linearized theory. The linearized theory can be improved to include results for the shock-shock by the method used in supersonic flow (see Whitham 1952). The improved solution has

$$\left.\begin{aligned} \frac{M-M_0}{M_0} &= \frac{F(\tau)}{\sqrt{(2Br)}}, \\ \theta &= \frac{BF(\tau)}{\sqrt{(2Br)}}, \end{aligned}\right\} \tag{67}$$

in place of (65), where $\tau(x, r)$ has to give a more accurate approximation to the true characteristic than the linear $\tau = x - Br$. On the characteristic curve τ = constant, we have from (29),

$$\frac{dx}{dr} = \cot(\theta + m) = \cot m_0 - \frac{1}{\sin^2 m_0}\left\{\theta + \left(\frac{dm}{dM}\right)_0 (M - M_0)\right\} + O\left\{\theta^2, \left(\frac{M-M_0}{M_0}\right)^2\right\}.$$

Substituting from (67) and using $B = \cot m_0$, we have

$$\frac{dx}{dr} = B - \tfrac{1}{2}kF(\tau)\, r^{-\frac{1}{2}} + O\left(\frac{F^2}{r}\right),$$

where
$$k = (2B)^{\frac{1}{2}}(B^2+1)\left\{1 + \frac{M_0}{B}\left(\frac{dm}{dM}\right)_{M=M_0}\right\}. \tag{68}$$

Hence, the improved approximation for τ is

$$x = Br - kF(\tau)\, r^{\frac{1}{4}} + \tau. \tag{69}$$

The shock-shock can be determined from the characteristics (69) exactly as in the supersonic flow problem. At the shock-shock η and r are related by

$$r^{\frac{1}{4}} = \frac{2}{kF^2(\tau)} \int_0^\tau F(\tau')\, d\tau'. \tag{70}$$

Together with (69), this gives the shock-shock curve; (67) gives the values M_1 and θ_1 at the shock-shock.

To supplement the results of the last section we may specialize the results of this small perturbation theory to the case of a slender cone. If the cone angle is θ_w, then $S(x) = \pi\theta_w^2 x^2$. The linearized solution (63) becomes

$$\alpha' = x\theta_w^2 \left\{ \frac{\sqrt{[(x/Br)^2 - 1]}}{x/Br} - \cosh^{-1}(x/Br) \right\}; \tag{71}$$

hence,

$$\left.\begin{aligned} \alpha'_x &= -\theta_w^2 \cosh^{-1}(x/Br), \\ \alpha'_r &= \theta_w^2 B \sqrt{[(x/Br)^2 - 1]}. \end{aligned}\right\} \tag{72}$$

These quantities vanish at $x/Br = 1$, and become

$$\begin{aligned} \alpha'_x &= -\theta_w^2 \log \frac{2}{B\theta_w}, \\ \alpha'_r &= \theta_w, \end{aligned}$$

at the cone where $x/Br \sim 1/B\theta_w$. Therefore, according to (64), the Mach number at the surface of the cone is given by

$$\frac{M_w - M_0}{M_0} = \left(\theta_w^2 \log \frac{2}{B\theta_w} - \tfrac{1}{2}\theta_w^2\right) + O\left(\theta_w^4 \log^2 \frac{1}{\theta_w}\right). \tag{73}$$

In the improved theory near the shock-shock, $F(\tau) = 2\theta_w^2 \tau^{\frac{1}{2}}$; hence, (70) gives

$$\tau = \frac{9k^2\theta_w^4 r}{4},$$

and the shock-shock is the straight line

$$x = (B - \tfrac{3}{4}k^2\theta_w^4)\, r.$$

The shock-shock angle χ is

$$\chi = m_0 + \tfrac{3}{2}B(B^2+1) \left\{ 1 + \frac{M_0}{B} \left(\frac{dm}{dM} \right)_0 \right\}^2 \theta_w^4, \tag{74}$$

and the values of θ and M at the shock-shock are

$$\theta_1 = 3B(B^2+1) \left\{ 1 + \frac{M_0}{B} \left(\frac{dm}{dM} \right)_0 \right\} \theta_w^4, \tag{75}$$

$$\frac{M_1 - M_0}{M_0} = 3(B^2+1) \left\{ 1 + \frac{M_0}{B} \left(\frac{dm}{dM} \right)_0 \right\} \theta_w^4. \tag{76}$$

If M_0 is large, $m_0 \sim 23{\cdot}9^\circ$, $B \sim 2{\cdot}253$ and $M_0\,(dm/dM)_0 \sim 0$ (see (48)). The results for this case have been given in (61) and (62).

(ii) *Stability of a plane shock wave*

In the stability problem, the initial position and Mach number distribution of the shock wave are given, and the problem is to determine the subsequent motion. Here, it is assumed that the initial position is

$$x = -f(y, z)$$

and the initial strength is $$M = M_0\{1 - g(y, z)\},$$

where f and g are small. Then, with

$$\alpha = M_0(x + \alpha'),$$

we have (using (17)) $$\alpha' = f(y, z), \quad \alpha'_x = g(y, z),$$

initially. Strictly speaking, these are the values on $x = -f(y, z)$, but it is consistent with the linearization already used to apply these initial conditions at $x = 0$. The solution of

$$B^2 \alpha'_{xx} = \alpha'_{yy} + \alpha'_{zz}$$

must be found subject to these initial values.

The analogy with the two-dimensional initial value problem of acoustics is obvious, and the solution is well known. It can be written as

$$\alpha' = \frac{B^2}{2\pi}\left\{\frac{\partial}{\partial x}\iint \frac{f(\eta, \zeta)\, d\eta\, d\zeta}{\sqrt{[x^2 - B^2(\eta - y)^2 - B^2(\zeta - z)^2]}} + \iint \frac{g(\eta, \zeta)\, d\eta\, d\zeta}{\sqrt{[x^2 - B^2(\eta - y)^2 - B^2(\zeta - z)^2]}}\right\}, \tag{77}$$

where the integrations are over the interior of the circle

$$B^2(\eta - y)^2 + B^2(\zeta - z)^2 = x^2.$$

In general, the disturbance given by (77) decays as it spreads out on the shock wave; if the disturbance is initially confined to a finite region, the decay is ultimately like $x^{-\frac{1}{2}}$ for large x. For the two-dimensional disturbance discussed in Part 1 there is no decay according to the linearized theory. But certain non-linear features, related to the formation of shock-shocks become crucially important after a sufficient time. When they were included in the two-dimensional problem a decay like $x^{-\frac{1}{2}}$ was found (see Part 1). Similar effects arise here and modify the ultimate decay law for an initially finite region from $\alpha' \propto x^{-\frac{1}{2}}$ to $\alpha' \propto x^{-\frac{3}{4}}$ for large x. No details will be given, since a very full treatment for the closely similar problems of sound waves has been presented in a previous paper (Whitham 1956). The linearized expression (77) should give a good approximation in general until x becomes quite large.

4. Further applications

As explained above, many problems could be posed and solved for the small perturbation theory simply by re-interpreting the large number of solutions available in linearized supersonic flow and acoustics. Two of the simpler examples have been given in § 3, and others can be worked out as required. Generally speaking, however, these solutions for small disturbances will probably be of less practical value here than in supersonic flow or acoustics, and the main emphasis should be on the solution of the full non-linear equations. For axi-symmetrical problems the numerical solution of quite complicated cases would be straightforward. But even for the full three-dimensional problems, numerical methods become feasible in the approximate theory of this paper, since there are only three independent variables x, y, z. (This would seem to be out of the question in the exact formulation which involves all four variables x, y, z and t.) Furthermore, there are special cases other than axi-symmetrical problems, where similarity solutions apply and the three variables can be reduced to two. For example, in the diffraction of a plane shock wave by a flat plate delta wing at incidence, the solution must take the form

$$\alpha = xf\left(\frac{y}{x}, \quad \frac{z}{x}\right),$$

since there is no fundamental length prescribed in the problem. This type of example corresponds to cone field theory in supersonic flow.

For genuine three-variable problems, the theory developed in § 1 leads to partial differential equations which are very similar to those in the *exact* formulation of *two-dimensional* time-dependent gas dynamics and numerical methods devised for such problems could be used here.

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